ALGEBRAIC CURVES
EXERCISE SHEET 1

Exercise 1.

Let R be a ring and $I \subsetneq R$ a proper ideal of R. A ring is called *reduced* if it does not contain any (non-zero) nilpotent element.

(1) Show there is a bijection between the ideals of R/I and the ideals of R containing I.
(2) Show that I is maximal if, and only if, R/I is a field.
(3) Show that I is prime if, and only if, R/I is a domain.
(4) Show that I is radical if, and only if, R/I is reduced.

Exercise 2.

Let R be a ring, S a multiplicative subset of R and M an R-module. Define the localization of M as the set $S^{-1}M = \{\frac{m}{s},\ m \in M,\ s \in S\}/\sim$, where : $\frac{m}{s} \sim \frac{m'}{s'} \Leftrightarrow \exists t \in S,\ t \cdot (s' \cdot m - s \cdot m') = 0$

(1) Check that the relation $\sim$ defined above is indeed an equivalence relation and that $S^{-1}M$ is an $S^{-1}R$-module.
(2) Show that localization preserves short exact sequences i.e. if
$$0 \to L \to M \to N \to 0$$
is an exact sequence of R-modules, then
$$0 \to S^{-1}L \to S^{-1}M \to S^{-1}N \to 0$$
is an exact sequence of $S^{-1}R$-modules.
(3) Let $I \subset R$ be an ideal of R. Show that $S^{-1}I$ is an ideal of $S^{-1}R$ and that we have an isomorphism of rings
$$S^{-1}R/S^{-1}I \simeq (S/I)^{-1}(R/I),$$
where S/I denotes the image of S in R/I.

Exercise 3.

Show that, for any exact sequence of finite-dimensional vector spaces :
$$0 \to V_1 \to V_2 \to \ldots \to V_n \to 0$$

the following relation holds :

$$\sum_{i=1}^{n}(-1)^i \cdot dim(V_i) = 0$$

Exercise 4.

(1) Let k be an infinite field, $F \in k[X_1, ..., X_n]$. Suppose $F(a_1, ..., a_n) = 0$ for all $a_1, ..., a_n \in k$. Show that $F = 0$.

(2) Give an example showing that a similar result doesn't hold for a finite field.

Exercise 5.

Let $F \in k[X_1, \ldots, X_n]$ be homogeneous of degree d. Show that :

$$\sum_{i=1}^{n} X_i \cdot \frac{\partial F}{\partial X_i} = d \cdot F$$